STEP I 2009 Solutions and Mark Scheme

Principal Examiner

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Contents

Foreword

This document contains model solutions to the 2009 STEP Mathematics Paper I. The solutions are fully worked and contain more detail and explanation than would be expected from candidates. They are intended to help students understand how to answer the questions, and therefore they are encouraged to attempt them first before looking at these model answers.

To further help them, there are various pieces of Commentary, which are written in a more conversational style, and are intended to explain some of the thinking behind a particular choice of approach.

This document also contains a Mark Scheme. This was used by the examiners during the marking process. It is important to remember that the nature of these questions is such that there may be multiple acceptable ways of answering them. As in any examination, the mark scheme was adapted appropriately for these alternative approaches; these adaptations are not recorded here.

The meanings of the marks are as in the standard GCSE and AS/A2 mark schemes:

- M marks for method
- A marks for correct answers, dependent on gaining the corresponding M mark(s)
- B marks are independent accuracy marks
- (ft) means that incorrect working is followed through
- (dep) means this mark is dependent upon gaining the previous mark
- cao/cso means 'correct answer/solution only'

Question 1

> *A* proper factor *of an integer N is a positive integer, not 1 or N, that divides N.*
>
> *(i) Show that $3^2 \times 5^3$ has exactly 10 proper factors.*

It is not by accident that this question writes "$3^2 \times 5^3$" and not "1125": it is aiming to suggest that it is much more straightforward to think about factors of a number if we are given its prime factorisation to begin with. Also, note that the question does not ask us to multiply out the factorisations at any point. In fact, there is no need to even give the factors explicitly if you do not need to.

Determining the proper factors of $3^2 \times 5^3$ is straightforward: any factor must be of the form $3^r \times 5^s$ with $0 \leqslant r \leqslant 2$ and $0 \leqslant s \leqslant 3$, giving the factors:

$$\begin{array}{ll}
3^0 \times 5^0 \ (= 1) & \text{(this is not a proper factor)} \\
3^0 \times 5^1 \ (= 5) & \\
3^0 \times 5^2 \ (= 25) & \\
3^0 \times 5^3 \ (= 125) & \\
3^1 \times 5^0 \ (= 3) & \\
3^1 \times 5^1 \ (= 15) & \\
3^1 \times 5^2 \ (= 75) & \\
3^1 \times 5^3 \ (= 375) & \\
3^2 \times 5^0 \ (= 9) & \\
3^2 \times 5^1 \ (= 45) & \\
3^2 \times 5^2 \ (= 225) & \\
3^2 \times 5^3 \ (= 1125) & \text{(this is not a proper factor)}
\end{array}$$

Therefore there are 10 proper factors in total.

Alternatively, we could simply note that there are 3 possible values for the power of 3 (namely 0, 1 and 2) and 4 for the power of 5 (namely 0, 1, 2 and 3), making $3 \times 4 = 12$ factors. Of these, two are not proper (1 and the number $3^2 \times 5^3$ itself), leaving $12 - 2 = 10$ proper factors.

Marks

M1: Listing factors of $3^2 \times 5^3$ in a systematic fashion

M1: Rejecting the cases $3^0 \times 5^0$ and $3^2 \times 5^3$

A1 cso: Must clearly demonstrate that all the factors have been found and none have been forgotten

Alternative approach:

M1: Identifying that factors are of the form $3^r \times 5^s$ with ranges (not necessarily correct) for the values of r and s

M1: Counting the cases as 3×4 with justification
A1 cso: Justify the exclusion of 2 cases

(i) (cont.)

Show that $3^2 \times 5^3$ has exactly 10 proper factors. Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors.

Now that we have done this and understood how to count the factors of $3^2 \times 5^3$, we can answer the second part: the number of proper factors of $3^a \times 5^b$ is $(a+1)(b+1) - 2$, as the power of 3 in a factor can be 0, 1, ..., a, and the power of 5 can be 0, 1, ..., b. So we require $(a+1)(b+1) - 2 = 10$, or $(a+1)(b+1) = 12$. Here are the possibilities:

$a+1$	$b+1$	a	b	$n = 3^a \times 5^b$
1	12	0	11	$3^0 \times 5^{11}$
2	6	1	5	$3^1 \times 5^5$
3	4	2	3	$3^2 \times 5^3$
4	3	3	2	$3^3 \times 5^2$
6	2	5	1	$3^5 \times 5^1$
12	1	11	0	$3^{11} \times 5^0$

so there are 6 possibilities in total. This means that there are 5 other integers with the required properties.

We use these same ideas in part (ii).

Marks

M1: Counting factors of $3^a \times 5^b$ as $(a+1)(b+1)$ or ab or similar
A1 cao: Correct counting of proper factors
M1: For deducing $(a+1)(b+1) = 12$ or equivalent (follow through incorrect count of proper factors, as long as 2 has been subtracted from the whole count)
M1: Listing possibilities for $a+1$ and $b+1$, and deducing for a and b (at least four possibilities needed)
A1: All correct possibilities for $a+1$ and $b+1$
A1 cso: Need to justify somehow that each possibility for $a+1$ and $b+1$ is valid; allow the answer of 6 solutions (i.e., don't penalise for not saying "5 other integers")

(ii) Let N be the smallest positive integer that has exactly 426 proper factors. Determine N, giving your answer in terms of its prime factors.

Following the same ideas as in part (i), let $n = 2^a \times 3^b \times 5^c \times 7^d \times \cdots$ be the prime factorisation of the positive integer n. (Note that we should use a letter other than N to distinguish our arbitrary integer from the special one that we seek.)

Then the number of factors of n is $(a+1)(b+1)(c+1)(d+1)\cdots$, and we must subtract 2 to get the number of proper factors. Assuming now that n has 426 proper factors, we must have

$$(a+1)(b+1)(c+1)(d+1)\cdots - 2 = 426,$$

so

$$(a+1)(b+1)(c+1)(d+1)\cdots = 428.$$

Now we can factorise $428 = 2^2 \times 107$, and 107 is prime. So the possible factorisations of 428 are $428 = 2 \times 214 = 4 \times 107 = 2 \times 2 \times 107$, so there can be at most three prime factors in n. We are seeking the smallest such n, so we choose the smallest possible primes, giving the smaller ones higher powers and larger ones smaller powers. So the smallest values of n for each possible factorisation of 428 are as follows:

With $428 = 428$:	$n = 2^{427}$
With $428 = 2 \times 214$:	$n = 2^{213} \times 3$
With $428 = 4 \times 107$:	$n = 2^{106} \times 3^3$
With $428 = 2 \times 2 \times 107$:	$n = 2^{106} \times 3 \times 5$

Since we seek the smallest possible value, our answer is clearly $2^{106} \times 3 \times 5$, as $2^{107} \times 3 > 3^3 = 27 > 3 \times 5 = 15$.

Marks

*SC: If attempt to factorise 426 instead of 428, getting $426 = 2 \times 3 \times 71$, then candidate can gain those marks indicated with * below.*

M1: Writing n or N as a general product of prime factors (may be implied by subsequent working or not fully general)*

M1: Counting factors of n in terms of prime factorisation (again, may not necessarily be fully general)*

A1: Deducing 428 factors in total (may be implied by subsequent working)

M1: Determining possible factorisations of 428 (at least two of them)*

M1: Justifying or mentioning explicitly choosing smallest primes for n*

M1: Justifying or mentioning explicitly allocating smaller primes to higher powers*

M1 dep: For attempting to explicitly determine smallest n for given factorisations of 428 based on explicit correct reasoning*

Either:

A1: For correct smallest n for at least three factorisations of 428*

A1: For correct smallest n for all four factorisations of 428 (SC: all five factorisations of 426, being $426 \to 2^{425}$, $2 \times 213 \to 2^{212} \times 3$, $3 \times 142 \to 2^{141} \times 3^2$, $2 \times 3 \times 71 \to 2^{71} \times 3^2 \times 5$, $6 \times 71 \to 2^{71} \times 3^5$)*

Or:

A1: For correct smallest n for factorisations 4×107 and $2^2 \times 107$ accompanied by attempted justification of why $428 = 2 \times 214$ and $428 = 428$ will not give smallest n (SC: same for the factorisations 6×71 and $2 \times 3 \times 71$)*

A1: Correct justification of this assertion*

M1: Justification of why the chosen value of n is the smallest*

A1 cao: Correct answer $N = 2^{106} \times 3 \times 5$

Question 2

> *A curve has the equation*
> $$y^3 = x^3 + a^3 + b^3\,,$$
> *where a and b are positive constants. Show that the tangent to the curve at the point $(-a, b)$ is*
> $$b^2y - a^2x = a^3 + b^3\,.$$

Differentiating the equation of the curve with respect to x gives

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2.$$

Substituting $x = -a$ and $y = b$ gives $3b^2\,\frac{\mathrm{d}y}{\mathrm{d}x} = 3a^2$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = a^2/b^2$. Then the standard equation of a straight line gives

$$y - b = \frac{a^2}{b^2}(x + a),$$

which easily rearranges into the form $b^2y - a^2x = a^3 + b^3$, as required.

Marks
M1: Reasonable attempt at differentiation wrt x
A1: Correct differentiation
M1: Determination of derivative at $(-a, b)$; must find explicitly in terms of a and b to get this and the subsequent marks in this part of the question.
M1 dep: Substitution into equation for straight line; must show an essentially complete correct method for determining the equation of a straight line, even if there are errors in the algebra
A1 cso: Correctly determining the required form from the above

> *In the case $a = 1$ and $b = 2$, show that the x-coordinates of the points where the tangent meets the curve satisfy*
> $$7x^3 - 3x^2 - 27x - 17 = 0\,.$$

In the case $a = 1$, $b = 2$, the curve has equation $y^3 = x^3 + 9$, and the tangent at $(-1, 2)$ has equation $4y - x = 9$. We therefore substitute $4y = x + 9$ into $y^3 = x^3 + 9$ as follows (after multiplying by 4^3):

$$\begin{aligned}
& 64y^3 = 64x^3 + 576 \\
\Longrightarrow\quad & (x+9)^3 = 64x^3 + 576 \\
\Longrightarrow\quad & x^3 + 27x^2 + 243x + 729 = 64x^3 + 576 \\
\Longrightarrow\quad & 63x^3 - 27x^2 - 243x - 153 = 0 \\
\Longrightarrow\quad & 7x^3 - 3x^2 - 27x - 17 = 0, \qquad \text{on dividing by 9,}
\end{aligned}$$

and this is the equation required.

Marks

B1: Explicit equation of curve

B1: Explicit equation of tangent

M1: Substitution of tangent into curve to find intersections

M1: Expanding and simplifying the equation in a useful manner towards the desired result

A1 cso: Correct deduction of desired result

> *Hence find positive integers p, q, r and s such that*
> $$p^3 = q^3 + r^3 + s^3 \, .$$

Now our equation looks hard to solve, but we know that there is a solution at $x = -1$, as the curve and line are tangent at this point. In fact, since they are tangent, $x = -1$ must be a double root. So we can take out a factor of $(x+1)^2$ to get

$$\begin{aligned} & (x+1)(7x^2 - 10x - 17) = 0 \\ \implies & \qquad (x+1)^2(7x - 17) = 0. \end{aligned}$$

Thus either $x = -1$, which we aleady know, or $x = \frac{17}{7}$. Since this point lies on the line $4y - x = 9$, the y-coordinate is $\left(\frac{17}{7} + 9\right)/4 = \frac{20}{7}$. Thus, as this point also lies on the curve $y^3 = x^3 + a^3 + b^3$, we have

$$\left(\tfrac{20}{7}\right)^3 = \left(\tfrac{17}{7}\right)^3 + 1^3 + 2^3.$$

Now multiplying both sides by 7^3 gives us our required result:

$$20^3 = 17^3 + 7^3 + 14^3,$$

so a solution is $p = 20$, $q = 17$, $r = 7$, $s = 14$.

Marks

[Note: To gain any marks on this part of the question, the solution must *use the preceding results.]*

B1: Noting or otherwise deducing that $x = -1$ is a solution

M1: Taking out a factor of $(x+1)$

M1: Taking out a second factor of $(x+1)$

A1: Correct factorisation

A1 cao: Determination of $x \neq -1$

Alternative for the final M1 A1 A1:

M1: Apply the quadratic formula to the quadratic factor

A1: Correct simplification of surd

A1 cao: Determination of $x \neq -1$

M1: Determination of y-coordinate at x-coordinate found (dependent on reasonable attempt to find x-coordinate, so at least previous M1 awarded)

A1 ft: Correct y-coordinate (follow through incorrect x-coordinate, $x \neq -1$)

M1: Substitute into $y^3 = x^3 + a^3 + b^3$ with a and b written out explicitly
M1: Multiplying through by (denominator)^3 to gain integers
A1 cao: Correct solution to question, either the specified one or an integer multiple thereof; must be deduced from earlier results

Question 3

> (i) *By considering the equation $x^2 + x - a = 0$, show that the equation $x = (a-x)^{\frac{1}{2}}$ has one real solution when $a \geqslant 0$ and no real solutions when $a < 0$.*

This looks somewhat confusing at first glance; why might $x = (a-x)^{\frac{1}{2}}$, which can be rearranged as the given quadratic, only have one solution whereas the quadratic can have two? But we must remember that this equation involves a square root, and by convention, this is the positive square root; therefore, real solutions must satisfy both $x \geqslant 0$ and $a - x \geqslant 0$, even if the quadratic has other solutions in addition.

Consider the equation

$$x = (a-x)^{\frac{1}{2}}. \qquad (*)$$

If this is true, then squaring gives $x^2 = a - x$, or $x^2 + x - a = 0$. The solutions of this quadratic are given by

$$x = \frac{-1 \pm \sqrt{1+4a}}{2},$$

and these will be real solutions of $(*)$ if and only if $x \geqslant 0$ and $a - x \geqslant 0$, that is $0 \leqslant x \leqslant a$. But as $a - x = x^2$, we will always have $a - x \geqslant 0$ for real solutions of the quadratic, so we need only check that $x \geqslant 0$. For a solution to $(*)$, we therefore require the plus sign in the quadratic formula, and we also need $1 + 4a \geqslant 1$, so $a \geqslant 0$.

Thus, for $a < 0$, there are no real solutions to $(*)$, and for $a \geqslant 0$, $x = \big(-1 + \sqrt{1+4a}\big)/2$ is the unique real solution.

Marks

M1: Deducing the quadratic from $(*)$

M1 (indep): Solving the (given) quadratic using the quadratic formula or otherwise

M1: For both conditions on x ($x \geqslant 0$, $a - x \geqslant 0$)

M1: For deducing that $a - x \geqslant 0$ for all real solutions to the quadratic, or equivalent argument that $a - x \geqslant 0$ is satisfied for the solutions found

M1: Deducing the sign needed in the quadratic formula, or equivalent argument

A1: Deducing $a \geqslant 0$ needed for real solution

A1 cso: For correctly deduced conclusion (does not require explicit solution to equation)

> (i) *(cont.)*
>
> *Find the number of distinct real solutions of the equation*
>
> $$x = \big((1+a)x - a\big)^{\frac{1}{3}}$$
>
> *in the cases that arise according to the value of a.*

Since cube-rooting is invertible, we have

$$x = \big((1+a)x - a\big)^{\frac{1}{3}} \iff x^3 = (1+a)x - a.$$

We are thus trying to solve the cubic equation $x^3 - (1+a)x + a = 0$. Inspection reveals one root, $x = 1$, so we can factorise the cubic as $(x-1)(x^2 + x - a) = 0$. Using the discriminant of the quadratic factor, $1 + 4a$, we find that $x^2 + x - a = 0$ has 0, 1 or 2 real roots according to whether $1 + 4a < 0$, $1 + 4a = 0$ or $1 + 4a > 0$, respectively.

Hence the original equation has 1 real root if $a < -\frac{1}{4}$, 2 distinct real roots if $a = -\frac{1}{4}$ (being $x = 1$ and $x = -\frac{1}{2}$), and 3 real roots if $a > -\frac{1}{4}$.

In the latter case, there is the possibility that they are not all distinct, though, if $x = 1$ is a root of $x^2 + x - a = 0$. This only happens when $a = 2$, and in this case, there are also only 2 *distinct* real roots.

Marks

M1: Converting equation to cubic

B1: Identifying $x = 1$ as root of cubic

M1: Using discriminant on resulting quadratic to count distinct real roots of quadratic

A1 cao: Correct determination of the number of distinct real roots of quadratic

A1 cao: Deducing number of (not necessarily distinct) real roots of cubic

M1: Checking for root of quadratic equal to 1

A1 cao: Deducing only two distinct roots when $a = 2$

(ii) Find the number of distinct real solutions of the equation

$$x = (b+x)^{\frac{1}{2}}$$

in the cases that arise according to the value of b.

This is very similar to part (i), with the only difference being that this time we have $b + x$ rather than $a - x$. The argument should therefore be fairly similar to part (i).

Starting with the equation

$$x = (b+x)^{\frac{1}{2}}, \qquad (\dagger)$$

we again square this to get $x^2 = b + x$, or $x^2 - x - b = 0$.

From the first form, we see that to have any solutions, we must have $x \geqslant 0$ and $b + x \geqslant 0$. From the second, we see that the discriminant $1 + 4b \geqslant 0$ and $b + x = x^2 \geqslant 0$ as long as x is real. So if $b < -\frac{1}{4}$, there are no solutions.

The solutions to the quadratic are

$$x = \frac{1 \pm \sqrt{1+4b}}{2}.$$

In the case $b = -\frac{1}{4}$, the repeated solution is $x = \frac{1}{2} \geqslant 0$, so there is one solution in this case.

In the case $b > -\frac{1}{4}$, we still require $x \geqslant 0$ for solutions. The smaller root is

$$x = \frac{1 - \sqrt{1+4b}}{2},$$

which is negative if $b > 0$ and non-negative if $b \leqslant 0$.

Thus the equation (†) has no solutions if $b < -\frac{1}{4}$, one solution if $b = -\frac{1}{4}$ or $b > 0$, and two solutions if $-\frac{1}{4} < b \leqslant 0$.

It is instructive to sketch the curves $y = x$ and $y = (b+x)^{\frac{1}{2}}$ on the same axes to see why this should be the case.

Marks

M1: Squaring (†) to get quadratic

M1: Considering discriminant to count real roots

A1: Deducing no real solutions if $b < -\frac{1}{4}$

A1 cso: Exactly one real solution if $b = -\frac{1}{4}$ (need $x \geqslant 0$ explicitly mentioned or some other argument that there really is a solution for this mark)

M1: Consideration of sign of x in the case $b > -\frac{1}{4}$

A1 cao: Deduction of number of solutions in case $b > -\frac{1}{4}$ (split into cases $-\frac{1}{4} < b \leqslant 0$ and $b > 0$)

If the candidate has attempted to answer this part by pure graph-sketching, then mark as follows. If they have used a combined method, discuss with the Principal Examiner.

M1: Sketching $(b+x)^{\frac{1}{2}}$ for at least one non-zero value of b, getting general shape correct

A1: Indicating that the curve meets the x-axis at $x = -b$ or otherwise showing how the value of b relates to the position of the curve

M1: Deducing that there are two solutions for $b_0 < b \leqslant 0$, one solution for $b = b_0$ and $b > 0$, and no solutions for $b < b_0$, where b_0 is some negative number where $y = (b+x)^{\frac{1}{2}}$ and $y = x$ are tangent (award this mark if the candidate is most of the way to this conclusion)

M1: Attempting to solve $(b+x)^{\frac{1}{2}} = x$ by squaring to determine the value of b which gives a double root

M1: Considering determinant to count real roots

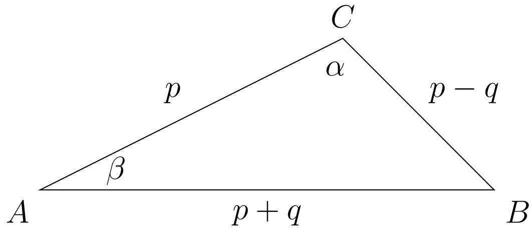
A1 cso: Determination of b_0 and correct conclusion

Question 4

The sides of a triangle have lengths $p-q$, p and $p+q$, where $p>q>0$. The largest and smallest angles of the triangle are α and β, respectively. Show by means of the cosine rule that

$$4(1-\cos\alpha)(1-\cos\beta)=\cos\alpha+\cos\beta.$$

In situations like this, it's often useful to draw a sketch to gain some clarity about what is happening. Note that the largest angle is always opposite the longest side, and the smallest angle is always opposite the shortest side.



Using the cosine rule with the angles at A and C gives, respectively:

$$(p+q)^2=p^2+(p-q)^2-2p(p-q)\cos\alpha \qquad (1)$$
$$(p-q)^2=p^2+(p+q)^2-2p(p+q)\cos\beta \qquad (2)$$

Then we need to manipulate these two equations in order to reach the desired result. There are several ways to do this; we show two of them.

Approach 1: Determining the cosines and substituting

Equation (1) gives, upon rearranging:

$$\begin{aligned}\cos\alpha&=\frac{p^2+(p-q)^2-(p+q)^2}{2p(p-q)}\\&=\frac{p^2+(p^2-2pq+q^2)-(p^2+2pq+q^2)}{2p(p-q)}\\&=\frac{p^2-4pq}{2p(p-q)}\\&=\frac{p-4q}{2(p-q)}. \qquad (3)\end{aligned}$$

Likewise, equation (2) yields:

$$\begin{aligned}\cos\beta &= \frac{p^2+(p+q)^2-(p-q)^2}{2p(p+q)} \\ &= \frac{p^2+(p^2+2pq+q^2)-(p^2-2pq+q^2)}{2p(p+q)} \\ &= \frac{p^2+4pq}{2p(p+q)} \\ &= \frac{p+4q}{2(p+q)}. \end{aligned} \tag{4}$$

Using equations (3) and (4), we now evaluate $4(1-\cos\alpha)(1-\cos\beta)$ and $\cos\alpha+\cos\beta$:

$$\begin{aligned}4(1-\cos\alpha)(1-\cos\beta) &= 4\left(1-\frac{p-4q}{2(p-q)}\right)\left(1-\frac{p+4q}{2(p+q)}\right) \\ &= 4.\frac{p+2q}{2(p-q)}.\frac{p-2q}{2(p+q)} \\ &= \frac{p^2-4q^2}{p^2-q^2}\end{aligned}$$

and

$$\begin{aligned}\cos\alpha+\cos\beta &= \frac{p-4q}{2(p-q)}+\frac{p+4q}{2(p+q)} \\ &= \frac{(p-4q)(p+q)+(p+4q)(p-q)}{2(p-q)(p+q)} \\ &= \frac{p^2-3pq-4q^2+p^2+3pq-4q^2}{2(p^2-q^2)} \\ &= \frac{2(p^2-4q^2)}{2(p^2-q^2)} \\ &= \frac{p^2-4q^2}{p^2-q^2}.\end{aligned}$$

Therefore we have the required equality

$$4(1-\cos\alpha)(1-\cos\beta)=\cos\alpha+\cos\beta. \tag{$*$}$$

Approach 2: Determining q/p and equating

From equation (1) above, we can expand to get

$$p^2+2pq+q^2=p^2+p^2-2pq+q^2-2p(p-q)\cos\alpha,$$

so that

$$p^2-4pq=2p(p-q)\cos\alpha.$$

We now divide by p^2 to get

$$1-4q/p=2(1-q/p)\cos\alpha.$$

We can now rearrange this to find q/p:

$$(2\cos\alpha - 4)q/p = 2\cos\alpha - 1,$$

so

$$\frac{q}{p} = \frac{2\cos\alpha - 1}{2\cos\alpha - 4}.$$

Doing the same for equation (2) gives:

$$p^2 - 2pq + q^2 = p^2 + p^2 + 2pq + q^2 - 2p(p+q)\cos\beta,$$

so that

$$p^2 + 4pq = 2p(p+q)\cos\beta.$$

Again, dividing by p^2 brings us to

$$1 + 4q/p = 2(1 + q/p)\cos\beta,$$

yielding

$$(4 - 2\cos\beta)q/p = 2\cos\beta - 1,$$

so

$$\frac{q}{p} = \frac{2\cos\beta - 1}{4 - 2\cos\beta}.$$

Equating these two expressions for q/p now gives us

$$\frac{2\cos\alpha - 1}{2\cos\alpha - 4} = \frac{2\cos\beta - 1}{4 - 2\cos\beta},$$

so that (cross-multiplying and dividing by two):

$$(2\cos\alpha - 1)(2 - \cos\beta) = (\cos\alpha - 2)(2\cos\beta - 1).$$

Now we expand the brackets to get

$$4\cos\alpha - 2 - 2\cos\alpha\cos\beta + \cos\beta = 2\cos\alpha\cos\beta - 4\cos\beta - \cos\alpha + 2,$$

so that

$$4 - 4\cos\alpha - 4\cos\beta + 4\cos\alpha\cos\beta = \cos\alpha + \cos\beta,$$

and the left hand side factorises to give us our desired result:

$$4(1 - \cos\alpha)(1 - \cos\beta) = \cos\alpha + \cos\beta.$$

Marks

NB: This question has multiple ways of answering it. Please adapt the mark scheme to match the approaches taken by candidates, and discuss these adaptations with the Principal Examiner before *you release the relevant scripts back into circulation.*

B1: Correct application of cosine rule to angle α (any valid form)

B1: Correct application of cosine rule to angle β (any valid form)

M1: Expanding or otherwise attempting to simplify an expression for $\cos\alpha$
A1: Correct simplification of $\cos\alpha$ *(award if unsimplified version correctly used later)*
M1: Likewise for $\cos\beta$
A1: Likewise for $\cos\beta$
M1: Evaluation and simplification of $4(1-\cos\alpha)(1-\cos\beta)$ *...*
A1: ... correctly
M1: Likewise for $\cos\alpha+\cos\beta$ *...*
A1: ... correctly
A1 cso: Deducing required equality

> *In the case* $\alpha = 2\beta$*, show that* $\cos\beta = \frac{3}{4}$ *and hence find the ratio of the lengths of the sides of the triangle.*

Substituting $\alpha = 2\beta$ into $(*)$ gives:

$$4(1-\cos 2\beta)(1-\cos\beta) = \cos 2\beta + \cos\beta.$$

We use the double angle formula for $\cos 2\beta$ to write this expression in terms of $\cos\beta$, giving:

$$4(2-2\cos^2\beta)(1-\cos\beta) = 2\cos^2\beta + \cos\beta - 1,$$

so

$$8(1+\cos\beta)(1-\cos\beta)^2 = (2\cos\beta - 1)(\cos\beta+1).$$

Since $\cos\beta \neq -1$, we can divide by $\cos\beta + 1$ to get

$$8(1-\cos\beta)^2 = 2\cos\beta - 1,$$

so we can rearrange to get

$$8\cos^2\beta - 18\cos\beta + 9 = 0,$$

which factorises as

$$(4\cos\beta - 3)(2\cos\beta - 3) = 0.$$

Since $\cos\beta = \frac{3}{2}$ is impossible, we must have $\cos\beta = \frac{3}{4}$, as required.

We now substitute this result into equation (2) to get

$$(p-q)^2 = p^2 + (p+q)^2 - 2p(p+q).\tfrac{3}{4}.$$

Expanding this gives

$$p^2 - 2pq + q^2 = p^2 + p^2 + 2pq + q^2 - \tfrac{3}{2}p^2 - \tfrac{3}{2}pq,$$

so

$$\tfrac{1}{2}p^2 - \tfrac{5}{2}pq = 0,$$

which gives $p = 5q$. Hence the side lengths are $p - q = 4q$, $p = 5q$ and $p + q = 6q$, which are in the ratio $4 : 5 : 6$.

An alternative way to do this last part is as follows. We have $\alpha = 2\beta$, so $\cos\alpha = 2\cos^2\beta - 1 = 2\cdot(\frac{3}{4})^2 - 1 = \frac{1}{8}$. It follows that $\sin\alpha = \frac{1}{8}\sqrt{63} = \frac{3}{8}\sqrt{7}$ and $\sin\beta = \frac{1}{4}\sqrt{7}$. We can now use the sine rule to get

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

or $a/c = \sin A/\sin C$. It follows that

$$\begin{aligned}\frac{p-q}{p+q} &= \frac{\sin\beta}{\sin\alpha}\\ &= \frac{\frac{1}{4}\sqrt{7}}{\frac{3}{8}\sqrt{7}}\\ &= \frac{2}{3},\end{aligned}$$

giving $3(p-q) = 2(p+q)$, or $p = 5q$. The rest of the result follows as above.

A third way of doing this, and arguably the simplest, is to substitute into equation (4), which gives:

$$\frac{3}{4} = \frac{p+4q}{2(p+q)}.$$

Multiplying both sides by $4(p+q)$ to clear the fractions gives

$$3(p+q) = 2(p+4q),$$

so that $p = 5q$. The rest of the argument again follows as above.

Marks
M1: Substituting $\alpha = 2\beta$ *into* $(*)$
M1: Using double angle formula for $\cos 2\beta$
M1: Dividing by or taking out factor of $\cos\beta + 1$
A1: Finding a quadratic for $\cos\beta$
M1: Factorising or otherwise solving quadratic to correctly find $\cos\beta$
M1: Substituting into (2)
M1: Rearranging to deduce quadratic for p *in terms of* q
A1: Finding side lengths in terms of p *or in terms of* q
A1 cao: Deducing ratio of side lengths

Alternative method for final four marks:
M1: Determine $\cos\alpha$ *and* $\sin\alpha$, $\sin\beta$
M1: Apply the sine rule to determine $(p-q)/(p+q)$ *or equivalent*
A1, A1 cao: as above

Third method for final four marks:
M1: Substitute into (4)
M1: Rearrange to find p *in terms of* q
A1, A1 cao: as above

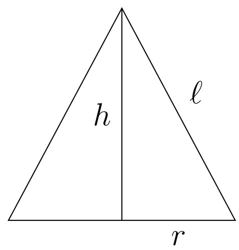
Question 5

A right circular cone has base radius r, height h and slant length ℓ. Its volume V, and the area A of its curved surface, are given by

$$V = \tfrac{1}{3}\pi r^2 h, \qquad A = \pi r \ell.$$

(i) Given that A is fixed and r is chosen so that V is at its stationary value, show that $A^2 = 3\pi^2 r^4$ and that $\ell = \sqrt{3}\,r$.

Since A is fixed and r is allowed to vary, we rearrange $A = \pi r \ell$ as $\ell = A/\pi r$. Also, we can draw a side view of the cone to determine the relationship between r, h and ℓ:



So clearly, $\ell^2 = h^2 + r^2$.

Substituting this into $\ell = A/\pi r$ gives

$$\ell^2 = h^2 + r^2 = \frac{A^2}{\pi^2 r^2},$$

which we can rearrange to give h^2 in terms of r and A:

$$h^2 = \frac{A^2}{\pi^2 r^2} - r^2.$$

Now $V = \frac{1}{3}\pi r^2 h$, so we have

$$\begin{aligned} V^2 &= \tfrac{1}{9}\pi^2 r^4 h^2 \\ &= \frac{\pi^2 r^4}{9}\left(\frac{A^2}{\pi^2 r^2} - r^2\right) \\ &= \tfrac{1}{9}(A^2 r^2 - \pi^2 r^6). \end{aligned}$$

(Working with V^2 rather than just V allows us to avoid square roots.)

Differentiating with respect to r gives

$$2V\frac{\mathrm{d}V}{\mathrm{d}r} = \tfrac{1}{9}(2A^2 r - 6\pi^2 r^5).$$

When V is at its stationary value, $\mathrm{d}V/\mathrm{d}r = 0$, so we require $2A^2 r - 6\pi^2 r^5 = 0$. As $r \neq 0$, we must have $6\pi^2 r^4 = 2A^2$, or $A^2 = 3\pi^2 r^4$, as wanted.

Substituting this into our formula for ℓ^2 gives

$$\begin{aligned}\ell^2 &= \frac{A^2}{\pi^2 r^2} \\ &= \frac{3\pi^2 r^4}{\pi^2 r^2} \\ &= 3r^2,\end{aligned}$$

so $\ell = \sqrt{3}\,r$, as we wanted to show.

Marks
B1: Relation between ℓ, h and r using Pythagoras
M1: Substituting to get h^2 in terms of A and r
A1 cao: Correct expression for h^2
M1: Using this to rewrite V^2 in terms of A and r
A1: Correct expression for V^2 (or for V)
M1: Reasonable attempt at differentiating V^2 (or V)
A1: Correct derivative
M1: Using $\mathrm{d}V/\mathrm{d}r = 0$ to find equation relating A^2 and r
A1 cso: Deducing required equation for A^2
M1: Substituting into formula for ℓ^2
A1 cso: Deducing required equation for ℓ

(ii) Given, instead, that V is fixed and r is chosen so that A is at its stationary value, find h in terms of r.

We have $\ell^2 = h^2 + r^2 = A^2/\pi^2 r^2$ and $V = \frac{1}{3}\pi r^2 h$. This time, V is fixed, so $h = 3V/\pi r^2$. Thus

$$\begin{aligned}A^2 &= \pi^2 r^2 (h^2 + r^2) \\ &= \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right) \\ &= \frac{9V^2}{r^2} + \pi^2 r^4\end{aligned}$$

Differentiating as before gives

$$2A\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{18V^2}{r^3} + 4\pi^2 r^3,$$

so $\mathrm{d}A/\mathrm{d}r = 0$ when $4\pi^2 r^6 = 18V^2$, so $2\pi r^3 = 3V\sqrt{2}$. Finally, substituting this into our formula $h = 3V/\pi r^2$ gives

$$\begin{aligned}h &= \frac{2\pi r^3/\sqrt{2}}{\pi r^2} \\ &= \sqrt{2}\,r.\end{aligned}$$

Marks

M1: Writing h in terms of V and r

M1: Substituting into formula for A^2

A1 cao: Correct formula for A^2 in terms of V and r

M1: Differentiating A^2 or A (reasonable attempt)

A1 cao: Correct derivative

M1: Solving $\mathrm{d}A/\mathrm{d}r = 0$ to get equation relating V and r

A1: Deducing $2\pi r^3 = 3V\sqrt{2}$ or equivalent

M1: Substituting into formula for h

A1 cao: Finding h in terms of r

Question 6

> *(i) Show that, for $m > 0$,*
>
> $$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x = \frac{(m-1)^3(m+1)}{2m^2} + \ln m\,.$$

We note that the numerator of the fraction (x^2) has a higher degree than the denominator ($x+1$), so we divide them first, getting $x^2 = (x+1)(x-1)+1$, so our integral becomes

$$\begin{aligned}
\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x &= \int_{1/m}^{m} x - 1 + \frac{1}{x+1}\,\mathrm{d}x \\
&= \Big[\tfrac{1}{2}x^2 - x + \ln|x+1|\Big]_{1/m}^{m} \\
&= \big(\tfrac{1}{2}m^2 - m + \ln|m+1|\big) - \big(\tfrac{1}{2}m^{-2} - m^{-1} + \ln|(1/m)+1|\big) \\
&= \frac{m^4 - 2m^3 - 1 + 2m}{2m^2} + \ln\frac{m+1}{(1/m)+1} \\
&= \frac{(m+1)(m^3 - 3m^2 + 3m - 1)}{2m^2} + \ln\frac{m(m+1)}{1+m} \\
&= \frac{(m+1)(m-1)^3}{2m^2} + \ln m.
\end{aligned}$$

(An alternative approach is to use the substitution $u = x+1$, which leads to exactly the same result.)

Marks

M1: Rewriting fraction as $x - 1 + \frac{1}{x+1}$ or using substitution $u = x+1$

A1 cao: Integrating integrand correctly (ignore limits and absence of absolute value signs for this mark)

M1: Substituting limits correctly and reasonable attempt at simplification

A1 cso: Deducing given expression for solution; must be clear how they reached $(m-1)^3(m+1)$ from the quartic expression

> *(ii) Show by means of a substitution that*
>
> $$\int_{1/m}^{m} \frac{1}{x^n(x+1)}\,\mathrm{d}x = \int_{1/m}^{m} \frac{u^{n-1}}{u+1}\,\mathrm{d}u\,.$$

Comparing the two integrals suggests that we should try the substitution $u = 1/x$. If we do this, we get $x = 1/u$ and $\mathrm{d}x/\mathrm{d}u = -1/u^2$. Also, the limits $x = 1/m$ and $x = m$ become

$u = m$ and $u = 1/m$ respectively. So we have

$$\begin{aligned}\int_{1/m}^{m} \frac{1}{x^n(x+1)}\,\mathrm{d}x &= \int_{m}^{1/m} \frac{1}{(1/u)^n(1/u+1)}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\ &= \int_{m}^{1/m} \frac{u^n}{1/u+1}\,\frac{-1}{u^2}\,\mathrm{d}u \\ &= -\int_{m}^{1/m} \frac{u^n}{u(1+u)}\,\mathrm{d}u \\ &= \int_{1/m}^{m} \frac{u^{n-1}}{u+1}\,\mathrm{d}u.\end{aligned}$$

Marks

M1: Substituting $u = 1/x$

M1 dep: Substituting correctly (condone incorrect $\mathrm{d}x/\mathrm{d}u$)

M1 dep: Reversing limits correctly (dependent on previous M1)

A1 cso: Fully correct solution

(iii) Evaluate:

(a) $$\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x\,.$$

This clearly relies on the earlier parts of the question, where $m = 2$. We can break the integral into two parts and use the results of (i) and (ii) as follows:

$$\begin{aligned}\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x &= \int_{1/2}^{2} \frac{x^5}{x^3(x+1)}\,\mathrm{d}x + 3\int_{1/2}^{2} \frac{1}{x^3(x+1)}\,\mathrm{d}x \\ &= \int_{1/2}^{2} \frac{x^2}{x+1}\,\mathrm{d}x + 3\int_{1/2}^{2} \frac{u^2}{u+1}\,\mathrm{d}u \\ &= 4\int_{1/2}^{2} \frac{x^2}{x+1}\,\mathrm{d}x \\ &= 4\left(\frac{(m+1)(m-1)^3}{2m^2} + \ln m\right) \\ &= 4(\tfrac{3}{8} + \ln 2) \\ &= \tfrac{3}{2} + 4\ln 2.\end{aligned}$$

Marks

M1: Splitting integral into two parts

M1: Simplifying first integral and applying (ii) to second integral

A1: Evaluating both integrals correctly (allow use of m instead of 2 for this mark)

M1: Substituting $m = 2$ in expression from (i) (this mark may be combined with preceding A1)
A1 cao: Purely numerical answer in given or equivalent form

> *(iii) Evaluate:*
>
> *(b)* $$\int_1^2 \frac{x^5 + x^3 + 1}{x^3(x+1)}\,\mathrm{d}x\,.$$

It is no longer so obvious how to proceed, as the limits are not of the form $1/m$ to m. So we break up the integral as in (a) and repeat the substitution of part (ii) once again, noting that the only change here is that the limits are different. We have:

$$\int_1^2 \frac{x^5}{x^3(x+1)}\,\mathrm{d}x = \int_1^2 \frac{x^2}{x+1}\,\mathrm{d}x,$$

and there is not much we can do at this point, short of evaluating this integral as in part (i).

Next, we have

$$\begin{aligned}\int_1^2 \frac{x^3}{x^3(x+1)}\,\mathrm{d}x &= \int_1^2 \frac{1}{x+1}\,\mathrm{d}x \\ &= \big[\ln|x+1|\big]_1^2 \\ &= \ln 3 - \ln 2.\end{aligned}$$

The third part gives us

$$\int_1^2 \frac{1}{x^3(x+1)}\,\mathrm{d}x = \int_{1/2}^1 \frac{u^2}{u+1}\,\mathrm{d}u,$$

by using the substitution of part (ii), but noting the the limits transform into $1/1 = 1$ and $1/2$, which are then reversed by the minus sign.

Adding all three terms and using part (i) with $m = 2$ now gives:

$$\begin{aligned}\int_1^2 \frac{x^5 + x^3 + 1}{x^3(x+1)}\,\mathrm{d}x &= \int_1^2 \frac{x^2}{x+1}\,\mathrm{d}x + \ln 3 - \ln 2 + \int_{1/2}^1 \frac{u^2}{u+1}\,\mathrm{d}u \\ &= \int_{1/2}^2 \frac{x^2}{x+1}\,\mathrm{d}x + \ln 3 - \ln 2 \\ &= \tfrac{3}{8} + \ln 2 + \ln 3 - \ln 2 \\ &= \tfrac{3}{8} + \ln 3.\end{aligned}$$

Marks
M1: Splitting up integral and simplifying all three parts
M1: Integrating $\frac{1}{x+1}$ mostly correctly
A1 cao: Integral $\int \frac{1}{x+1}\,\mathrm{d}x$ correctly evaluated
M1: Substituting $u = 1/x$ in third integral to get $\int \frac{u^2}{u+1}\,\mathrm{d}u$ (ignoring limits)

A1: Correct substitution, including limits
M1: Combining first and third integrals or evaluating both integrals
A1 cao: Evaluating all integrals and deducing correct final answer

Question 7

Show that, for any integer m,

$$\int_0^{2\pi} \mathrm{e}^x \cos mx \ \mathrm{d}x = \frac{1}{m^2+1}\left(\mathrm{e}^{2\pi}-1\right).$$

This is a standard integral with standard techniques for solving it. The approach used here is not the only possible one, but is fairly general.

We write $I = \int_0^{2\pi} \mathrm{e}^x \cos mx \ \mathrm{d}x$ and apply integration by parts twice, taking great care of the signs, as follows:

$$\begin{aligned}
I &= \int_0^{2\pi} \mathrm{e}^x \cos mx \ \mathrm{d}x \\
&= \left[\mathrm{e}^x \cdot \frac{1}{m}\sin mx\right]_0^{2\pi} - \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m}\sin mx \ \mathrm{d}x \\
&= \left(\tfrac{1}{m}\mathrm{e}^{2\pi}\sin 2\pi m\right) - \left(\tfrac{1}{m}\mathrm{e}^{0}\sin 0\right) - \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m}\sin mx \ \mathrm{d}x \\
&= 0 - \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m}\sin mx \ \mathrm{d}x \\
&= -\left[\mathrm{e}^x \cdot \frac{1}{m^2}(-\cos mx)\right]_0^{2\pi} + \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m^2}(-\cos mx) \ \mathrm{d}x \\
&= \left(\frac{1}{m^2}\mathrm{e}^{2\pi}\cos 2\pi m\right) - \left(\frac{1}{m^2}\mathrm{e}^{0}\cos 0\right) - \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m^2}\cos mx \ \mathrm{d}x \\
&= \frac{1}{m^2}\left(\mathrm{e}^{2\pi}-1\right) - \frac{1}{m^2}I.
\end{aligned}$$

Multiplying throughout by m^2 gives

$$m^2 I = \left(\mathrm{e}^{2\pi}-1\right) - I,$$

and now adding I to both sides and dividing by m^2+1 gives the desired result.

We performed these integrations by writing $\int \mathrm{e}^x \cos mx \ \mathrm{d}x$ in the form $\int v\frac{\mathrm{d}u}{\mathrm{d}x} \ \mathrm{d}x$, where $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos mx$ and $v = \mathrm{e}^x$. We could equally well have chosen $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ and $v = \cos mx$, and would have ended up with the same conclusion.

Marks
M1: Application of parts once
A1: Correct evaluation of integral (following first application of parts)
M1 dep: Second application of parts (dependent on first M1)
A1: Correct evaluation of integral (following second application of parts)
A1 cso: Deduction of stated result by rearranging

(i) Expand $\cos(A + B) + \cos(A - B)$*. Hence show that*

$$\int_0^{2\pi} \mathrm{e}^x \cos x \cos 6x \; \mathrm{d}x = \tfrac{19}{650}\big(\mathrm{e}^{2\pi} - 1\big)\,.$$

We have

$$\begin{aligned}\cos(A + B) + \cos(A - B) &= \cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B \\ &= 2 \cos A \cos B.\end{aligned}$$

Matching this to the integral we have been given, we set $A = 6x$ and $B = x$ (so that $A - B$ is positive, though this is not critical), giving

$$2 \cos x \cos 6x = \cos 7x + \cos 5x.$$

Thus our integral becomes

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^x \cos x \cos 6x \; \mathrm{d}x &= \frac{1}{2} \int_0^{2\pi} \mathrm{e}^x (\cos 7x + \cos 5x) \; \mathrm{d}x \\ &= \frac{1}{2} \left(\frac{1}{7^2 + 1}\big(\mathrm{e}^{2\pi} - 1\big) + \frac{1}{5^2 + 1}\big(\mathrm{e}^{2\pi} - 1\big) \right) \\ &= \frac{1}{2} \left(\frac{1}{50} + \frac{1}{26} \right) \big(\mathrm{e}^{2\pi} - 1\big) \\ &= \frac{1}{2} \cdot \frac{26 + 50}{1300} \big(\mathrm{e}^{2\pi} - 1\big) \\ &= \tfrac{19}{650}\big(\mathrm{e}^{2\pi} - 1\big),\end{aligned}$$

as required.

Marks

M1: Expanding the cosines using the compound angle formulæ
A1 cso: Deducing factor formula stated
M1: Setting $A = 6x$ *and* $B = x$ *or vice versa*
A1: Rewriting $2 \cos x \cos 6x$ *as* $\cos 7x + \cos 5x$ *or equivalent*
M1: Using (i) to evaluate integral
M1 dep: Factorising out $\mathrm{e}^{2\pi} - 1$ *and simplifying resulting fractions*
A1 cso: Deducing stated result

(ii) Evaluate $\displaystyle\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x \; \mathrm{d}x$.

We are clearly asked to do the same type of trick again. Here is one way to proceed.

We are looking for the product of two sines, and this appears in the compound angle formula for cosine. So we consider

$$\begin{aligned}\cos(A+B)-\cos(A-B) &= \cos A\cos B-\sin A\sin B-\cos A\cos B-\sin A\sin B\\ &= -2\sin A\sin B.\end{aligned}$$

We can therefore write $2\sin 2x\sin 4x = \cos 2x - \cos 6x$, using $A = 4x$ and $B = 2x$. This gives the integral as

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x \,\mathrm{d}x &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x(\cos 2x - \cos 6x)\cos x \,\mathrm{d}x\\ &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x \cos 2x\cos x \,\mathrm{d}x - \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x \cos 6x \cos x \,\mathrm{d}x.\end{aligned}$$

Now the second integral is exactly the one we evaluated in (i), and the first integral can be approached in the same way, with $A = 2x$ and $B = x$, so $2\cos 2x\cos x = \cos 3x + \cos x$. Therefore

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^x \cos 2x\cos x \,\mathrm{d}x &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x(\cos 3x + \cos x)\,\mathrm{d}x\\ &= \frac{1}{2}\left(\frac{1}{3^2+1}\big(\mathrm{e}^{2\pi}-1\big) + \frac{1}{1^2+1}\big(\mathrm{e}^{2\pi}-1\big)\right)\\ &= \frac{1}{2}\left(\frac{1}{10}+\frac{1}{2}\right)\big(\mathrm{e}^{2\pi}-1\big)\\ &= \tfrac{3}{10}\big(\mathrm{e}^{2\pi}-1\big).\end{aligned}$$

Finally, subtracting the two integrals gives

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x \,\mathrm{d}x &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x \cos 2x\cos x \,\mathrm{d}x - \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x \cos 6x \cos x \,\mathrm{d}x\\ &= \tfrac{1}{2}\cdot\tfrac{3}{10}\big(\mathrm{e}^{2\pi}-1\big) - \tfrac{1}{2}\cdot\tfrac{19}{650}\big(\mathrm{e}^{2\pi}-1\big)\\ &= \tfrac{1}{2}\big(\tfrac{195}{650}-\tfrac{19}{650}\big)\big(\mathrm{e}^{2\pi}-1\big)\\ &= \tfrac{1}{2}\cdot\tfrac{176}{650}\big(\mathrm{e}^{2\pi}-1\big)\\ &= \tfrac{44}{325}\big(\mathrm{e}^{2\pi}-1\big).\end{aligned}$$

Marks

M1: Deducing another factor formula (either for $2\sin A\sin B$ or $2\sin A\cos B$)
M1 dep: Applying this to the integrand
A1: Splitting the integral into two simpler integrals
M1: Applying factor formula to resulting integral(s)
M1: Using (i) to evaluate resulting integrals
A1: For evaluating each $\int \mathrm{e}^x \cos ax \cos bx$ or the like
M1: Combining the various integrals to determine the original integral
A1 cao: Evaluating the original integral (fraction need not be simplified to lowest terms)

Question 8

> *(i) The equation of the circle C is*
>
> $$(x - 2t)^2 + (y - t)^2 = t^2,$$
>
> *where t is a positive number. Show that C touches the line $y = 0$.*

This is a circle with centre $(2t, t)$ and radius t, therefore it touches the x-axis, which is distance t from the centre.

Alternatively, we are looking to solve the simultaneous equations

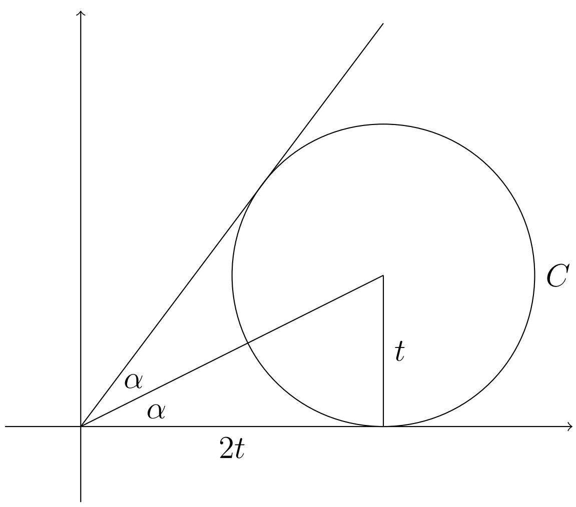
$$\begin{aligned} (x - 2t)^2 + (y - t)^2 &= t^2 \\ y &= 0. \end{aligned}$$

Substituting the second equation into the first gives $(x - 2t)^2 + t^2 = t^2$, or $(x - 2t)^2 = 0$. Since this only has one solution, $x = 2t$, the line must be tangent to the circle.

Marks
Either: *M1: Description of circle*
A1 cso: Conclusion
Or:
M1: Substituting $y = 0$ into equation of circle
A1 cso: Conclusion

> *(i) (cont.)*
>
> *Let α be the acute angle between the x-axis and the line joining the origin to the centre of C. Show that $\tan 2\alpha = \frac{4}{3}$ and deduce that C touches the line $3y = 4x$.*

We begin by drawing a sketch of the situation.



Clearly, therefore, $\tan\alpha = t/2t = \frac{1}{2}$, so

$$\begin{aligned}\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha} \\ &= \frac{1}{1-\frac{1}{4}} \\ &= \tfrac{4}{3}.\end{aligned}$$

From the sketch, it is clear that by symmetry, the line through the origin which makes an angle of 2α with the x-axis touches the circle C. Since the gradient of this line is $\tan 2\alpha = \frac{4}{3}$ and it passes through the origin, it has equation $y = \frac{4}{3}x$, or $3y = 4x$.

Marks

M1: Correct sketch of situation (may be implied by later working)

B1: Evaluating $\tan\alpha$

M1: Evaluating $\tan 2\alpha$ *using double angle formula*

A1 cso: Showing $\tan 2\alpha = \frac{4}{3}$

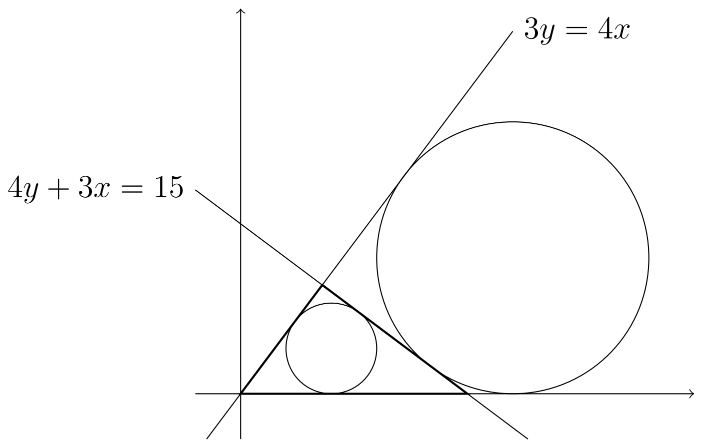
M1: Argument by symmetry or otherwise that line at angle 2α through origin touches C

A1 cso: Deducing equation of this line (need to mention gradient equals tan of angle or similar)

(ii) Find the equation of the incircle of the triangle formed by the lines $y = 0$, $3y = 4x$ and $4y + 3x = 15$.

Note: *The incircle of a triangle is the circle, lying totally inside the triangle, that touches all three sides.*

This circle has the properties described in part (i), in that it touches both the x-axis (i.e., the line $y = 0$) and the line $3y = 4x$, and it lies above the x-axis, so it must have the form given. The only remaining condition is that it must touch the line $4y + 3x = 15$. Two circles with centre $(2t, t)$ touch these three lines, but only one of them lies within the triangle, as illustrated on this sketch, so we must take the one with the smaller value of t:



Substituting $4y + 3x = 15$ into the equation of C will give us the intersections of the line and the circle. The line will be a tangent to C if and only if the discriminant of the resulting quadratic equation is 0. Doing this, from the equation of C:

$$(x - 2t)^2 + (y - t)^2 = t^2$$

we get:

$$(x - 2t)^2 + ((15 - 3x)/4 - t)^2 = t^2.$$

Multiplying both sides by 4^2 and expanding gives:

$$16x^2 - 64tx + 64t^2 + (15 - 3x)^2 - 8(15 - 3x)t + 16t^2 = 16t^2,$$

so

$$16x^2 - 64tx + 64t^2 + 225 - 90x + 9x^2 - 120t + 24tx + 16t^2 = 16t^2.$$

Collecting terms in x gives:

$$25x^2 - (90 + 40t)x + 225 - 120t + 64t^2 = 0.$$

Since this has a repeated root, as the line is required to be tangent to the circle, the discriminant must be zero, so

$$(90 + 40t)^2 - 4 \times 25(225 - 120t + 64t^2) = 0,$$

so

$$(9 + 4t)^2 - (225 - 120t + 64t^2) = 0,$$

or

$$16t^2 + 72t + 81 - (225 - 120t + 64t^2) = 0,$$

giving

$$-48t^2 + 192t - 144 = 0.$$

Dividing all the terms by -48 gives $t^2 - 4t + 3 = 0$, which factorises as $(t - 1)(t - 3) = 0$. Thus the two circles in the sketch are given by $t = 1$ and $t = 3$, and the incircle is clearly the one with $t = 1$. So the incircle has equation

$$(x - 2)^2 + (y - 1)^2 = 1.$$

Marks

M1: Circle must have the form given in part (i)

M1: Need to choose the smaller value of t to get the incircle rather than the excircle (must be justified at some point to get this mark and final A mark)

M1: Tangent if and only if discriminant is zero (condone "if" instead of "iff")

M1: Substituting $4y + 3x = 15$ into equation of C

M1 dep: Expanding equation with the aim of collecting like terms

M1 dep: Collecting terms in x^2, x and constant term

A1: Correct quadratic in x

M1: Finding the discriminant of this quadratic

M1: Rearranging to determine quadratic for t

A1: Correct quadratic for t

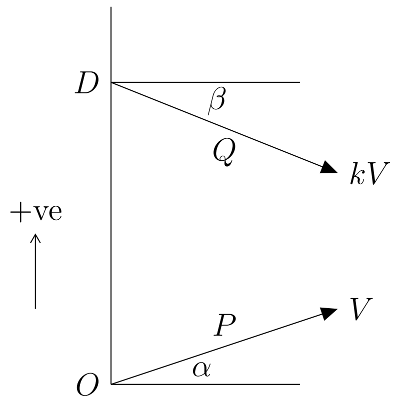
A1 cao: Solving quadratic to find t

A1 cso: Correct equation for circle (must have justified choice of t for this mark)

Question 9

> *Two particles P and Q are projected simultaneously from points O and D, respectively, where D is a distance d directly above O. The initial speed of P is V and its angle of projection above the horizontal is α. The initial speed of Q is kV, where $k > 1$, and its angle of projection below the horizontal is β. The particles collide at time T after projection.*
> *Show that $\cos\alpha = k\cos\beta$ and that T satisfies the equation*
>
> $$(k^2 - 1)V^2T^2 + 2dVT\sin\alpha - d^2 = 0.$$

We begin by drawing a sketch showing the initial situation.



For components, we will write u_P and u_Q for the horizontal components of the velocities of P and Q respectively, and v_P and v_Q for the vertical components (measured upwards). For the displacements from O, we write x_P and x_Q for the horizontal components and y_P and y_Q for the vertical components.

Resolving horizontally, using the "suvat" equations, we have

$$\begin{aligned} u_P &= V\cos\alpha, \\ u_Q &= kV\cos\beta; \\ x_P &= Vt\cos\alpha, \\ x_Q &= kVt\cos\beta. \end{aligned}$$

Likewise, vertically we have

$$\begin{aligned} v_P &= V\sin\alpha - gt, \\ v_Q &= -kV\sin\beta - gt; \\ y_P &= Vt\sin\alpha - \tfrac{1}{2}gt^2, \\ y_Q &= d - kVt\sin\beta - \tfrac{1}{2}gt^2. \end{aligned}$$

At time T, the particles collide, so $x_P = x_Q$, giving

$$VT\cos\alpha = kVT\cos\beta,$$

so that $\cos\alpha = k\cos\beta$.

Also, $y_P = y_Q$, so

$$VT\sin\alpha - \tfrac{1}{2}gT^2 = d - kVT\sin\beta - \tfrac{1}{2}gT^2,$$

which gives

$$VT\sin\alpha = d - kVT\sin\beta.$$

Now $k^2\sin^2\beta = k^2 - k^2\cos^2\beta = k^2 - \cos^2\alpha$ from the above, so we rearrange and then square both sides of the previous equation to make use of this conclusion:

$$kVT\sin\beta = d - VT\sin\alpha,$$

so

$$k^2V^2T^2\sin^2\beta = (d - VT\sin\alpha)^2,$$

which then gives us

$$V^2T^2(k^2 - \cos^2\alpha) = d^2 - 2dVT\sin\alpha + V^2T^2\sin^2\alpha.$$

Finally, subtracting the right hand side from the left gives

$$V^2T^2(k^2 - \cos^2\alpha - \sin^2\alpha) - d^2 + 2dVT\sin\alpha = 0,$$

or

$$V^2T^2(k^2 - 1) + 2dVT\sin\alpha - d^2 = 0, \qquad (*)$$

as required.

Marks

M1: Resolving velocities and/or displacements

M1: Use of "suvat" equations or equivalent

A1: Horizontal components of displacement for P and Q

A1: Vertical component of displacement for P

A1: Vertical component of displacement for Q (including distance above O, or explicitly measured from D)

M1: Equating horizontal components of displacement when collide

A1 cso: Deduction of $\cos\alpha = k\cos\beta$

M1: Equating vertical components of displacement when collide (must include d in equation)

B1: Expressing $k^2\sin^2\beta$ in terms of α (or equivalent)

M1: Rearranging vertical components equation to get $A\sin^2\beta = \cdots$

M1: Subtituting to eliminate β from equation

A1 cso: Deducing given quadratic in T

Given that the particles collide when P reaches its maximum height, find an expression for $\sin^2\alpha$ in terms of g, d, k and V, and deduce that

$$gd \leqslant (1+k)V^2.$$

At the maximum height, we have $v_P = 0$, so $V \sin\alpha = gT$. Substituting for T in $(*)$ gives us

$$V^2\left(\frac{V\sin\alpha}{g}\right)^2(k^2-1)+2dV\left(\frac{V\sin\alpha}{g}\right)\sin\alpha-d^2=0.$$

Multiplying through by g^2 and expanding brackets gives

$$(k^2-1)V^4\sin^2\alpha+2gdV^2\sin^2\alpha-g^2d^2=0.$$

Thus

$$\sin^2\alpha=\frac{g^2d^2}{(k^2-1)V^4+2gdV^2}.$$

For the inequality, the only thing we know for certain is that $\sin^2\alpha \leqslant 1$, and this gives

$$\frac{g^2d^2}{(k^2-1)V^4+2gdV^2}\leqslant 1,$$

so

$$g^2d^2\leqslant(k^2-1)V^4+2gdV^2.$$

It is not immediately clear how to continue, so we try completing the square for gd:

$$(gd-V^2)^2-V^4\leqslant(k^2-1)V^4,$$

so that

$$(gd-V^2)^2\leqslant k^2V^4.$$

Now if $a^2 \leqslant b^2$, then $a \leqslant |b|$, so

$$gd-V^2\leqslant|kV^2|.$$

But $kV^2 > 0$, so we are almost there:

$$gd-V^2\leqslant kV^2,$$

which finally gives us the required

$$gd\leqslant(1+k)V^2.$$

Marks

M1: At maximum height, vertical component of velocity is zero, or derivative of vertical component of displacement is zero

A1: Deducing $V\sin\alpha = gT$

M1: Substituting into $(*)$ and multiplying through by g^2

A1 cao: Expression for $\sin^2\alpha$

M1: Using $\sin^2\alpha \leqslant 1$ in resulting expression

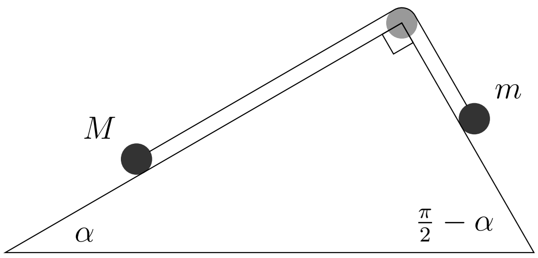
M1: Completing the square for gd

M1: Deducing $gd - V^2 \leqslant kV^2$ (condone ignoring absolute values issue for this mark)

A1 cso: Conclusion $gd \leqslant (1+k)V^2$; for this mark, need to note something about the need for $kV^2 \geqslant 0$

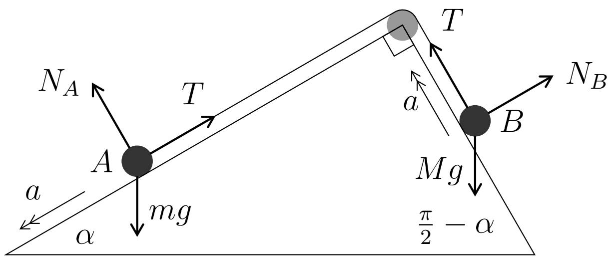
Question 10

A triangular wedge is fixed to a horizontal surface. The base angles of the wedge are α and $\frac{\pi}{2}-\alpha$. Two particles, of masses M and m, lie on different faces of the wedge, and are connected by a light inextensible string which passes over a smooth pulley at the apex of the wedge, as shown in the diagram. The contacts between the particles and the wedge are smooth.



(i) Show that if $\tan\alpha > \dfrac{m}{M}$ the particle of mass M will slide down the face of the wedge.

We start by drawing the forces on the picture, and labelling the masses as A (of mass M) and B (of mass m). We let T be the tension in the string.



We now resolve along the faces of the wedge at A and B. (It turns out that resolving normally to the face doesn't help at all for this problem.)

$$\mathscr{R}_A(\nearrow) \qquad T - Mg\sin\alpha = -Ma \tag{1}$$

$$\mathscr{R}_B(\nwarrow) \qquad T - mg\cos\alpha = ma \tag{2}$$

We are not interested in the tension in the string, so we subtract these equations (as $(2)-(1)$) to eliminate T, yielding

$$Mg\sin\alpha - mg\cos\alpha = ma + Ma.$$

Thus, dividing by $M+m$ gives

$$a = \frac{Mg\sin\alpha - mg\cos\alpha}{M+m}. \tag{3}$$

The mass M will slide down the slope if and only if $a > 0$, that is if and only if

$$Mg\sin\alpha - mg\cos\alpha > 0.$$

Rearranging and dividing by $g\cos\alpha$ gives $M\tan\alpha > m$, or

$$\tan\alpha > \frac{m}{M}.$$

Marks

M1: Reasonable attempt at a force diagram (may be implied by later working, though this is unlikely)

A1: Either fully correct at A, fully correct at B or correct at both modulo one repeated error (e.g., forgetting normal reaction)

A1 cao: All forces in force diagram correct, and no incorrect forces present

M1: Resolving along the plane at A or B

A1: Correct resolution at A

A1: Correct resolution at B

M1: Eliminating T

A1 cao: Deducing a correct expression for the acceleration

M1: Using $a > 0$ to deduce condition for sliding

A1 cso: Rearranging to conclude stated result

> *(ii) Given that* $\tan\alpha = \dfrac{2m}{M}$*, show that the magnitude of the acceleration of the particles is*
> $$\frac{g\sin\alpha}{\tan\alpha + 2}$$
> *and that this is maximised at* $4m^3 = M^3$ *.*

We simply substitute $2m/M = \tan\alpha$ into our formula (3) for a to find the magnitude of the acceleration:

$$\begin{aligned} a &= \frac{Mg\sin\alpha - mg\cos\alpha}{M+m} \\ &= \frac{2g\sin\alpha - (2m/M)g\cos\alpha}{2 + (2m/M)} && \text{multiplying by } 2/M \\ &= \frac{2g\sin\alpha - g\tan\alpha\cos\alpha}{2+\tan\alpha} \\ &= \frac{2g\sin\alpha - g\sin\alpha}{2+\tan\alpha} \\ &= \frac{g\sin\alpha}{2+\tan\alpha} \end{aligned}$$

To maximise this with respect to α, we differentiate with respect to α (using the quotient rule) and solve $\mathrm{d}a/\mathrm{d}\alpha = 0$:

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}\alpha} &= \frac{(2+\tan\alpha)g\cos\alpha - \sec^2\alpha . g\sin\alpha}{(2+\tan\alpha)^2} \\ &= \frac{g(2\cos\alpha + \sin\alpha - \sec^2\alpha\sin\alpha)}{(2+\tan\alpha)^2} \\ &= 0 \end{aligned}$$

so we require

$$2\cos\alpha + \sin\alpha - \sec^2\alpha\sin\alpha = 0.$$

We can simplify this by dividing through by $\cos\alpha$, so that we are able to express everything in terms of $\tan\alpha$:

$$2 + \tan\alpha - \sec^2\alpha\tan\alpha = 0,$$

so

$$2 + \tan\alpha - (1 + \tan^2\alpha)\tan\alpha = 0,$$

so that $\tan^3\alpha = 2$. Remembering that $\tan\alpha = 2m/M$, we finally get

$$\frac{8m^3}{M^3} = 2,$$

which leads immediately to $M^3 = 4m^3$.

We finally need to ensure that this stationary point gives us a maximum; this is clear, since as $\alpha \to 0$, $a \to 0$ and as $\alpha \to \frac{\pi}{2}$, $a \to 0$.

Marks

M1: For substituting $\tan\alpha = 2m/M$ *into the expression for acceleration*

M1: Reasonable manipulation of the resulting expression towards the required form

A1 cso: Reaching the stated form

M1: Differentiating a *wrt* α *to find stationary point; must be a reasonable attempt using quotient rule or equivalent*

A1: Correct derivative, even if not simplified

M1 (dep on previous M1): Deducing $2\cos\alpha + \sin\alpha - \sec^2\alpha\sin\alpha = 0$ *or equivalent; follow through incorrect derivative for this mark*

M1: Rearranging equation to get an expression involving only $\tan\alpha$

A1 cao: Deducing $\tan^3\alpha = 2$

A1: Deducing $M^3 = 4m^3$

B1 (dependent on reasonable attempt at finding stationary point): argument that this stationary point is a maximum

Question 11

> *Two particles move on a smooth horizontal table and collide. The masses of the particles are m and M. Their velocities before the collision are $u\mathbf{i}$ and $v\mathbf{i}$, respectively, where $\mathbf{i}$ is a unit vector and $u > v$. Their velocities after the collision are $p\mathbf{i}$ and $q\mathbf{i}$, respectively. The coefficient of restitution between the two particles is e, where $e < 1$.*
>
> *(i) Show that the loss of kinetic energy due to the collision is*
>
> $$\tfrac{1}{2}m(u-p)(u-v)(1-e),$$
>
> *and deduce that $u \geqslant p$.*

Before the collision:

After the collision:

Conservation of momentum gives:

$$mu + Mv = mp + Mq. \tag{1}$$

Newton's Law of Restitution gives:

$$q - p = e(u - v). \tag{2}$$

Now the loss of kinetic energy due to the collision is

$$\begin{aligned} E &= \left(\tfrac{1}{2}mu^2 + \tfrac{1}{2}Mv^2\right) - \left(\tfrac{1}{2}mp^2 + \tfrac{1}{2}Mq^2\right) \\ &= \tfrac{1}{2}m(u^2 - p^2) + \tfrac{1}{2}M(v^2 - q^2) \\ &= \tfrac{1}{2}m(u-p)(u+p) + \tfrac{1}{2}M(v-q)(v+q). \end{aligned}$$

Now from (1), we get $M(v - q) = m(p - u)$, so we get

$$\begin{aligned} E &= \tfrac{1}{2}m(u-p)(u+p) + \tfrac{1}{2}m(p-u)(v+q) \\ &= \tfrac{1}{2}m(u-p)\big((u+p) - (v+q)\big) \\ &= \tfrac{1}{2}m(u-p)\big((u-v) - (q-p)\big) \\ &= \tfrac{1}{2}m(u-p)\big((u-v) - e(u-v)\big) \\ &= \tfrac{1}{2}m(u-p)(u-v)(1-e). \end{aligned}$$

Now, since the loss of energy cannot be negative, we have $E \geqslant 0$. But we are given that $e < 1$ and $u > v$, so we must have $u - p \geqslant 0$, or $u \geqslant p$.

Marks

B1 cao: Conservation of momentum equation

B1 cao: (Newton's) Law of Restitution equation

M1: Calculating loss of KE

A1 cao: Simplified expression for loss of KE (need not break up difference of two squares for this)

M1: Use of CoM result to simplify loss of KE

M1 (indep): Use of LoR result to simplify loss of KE

A1 cso: Deducing stated formula for loss of KE

B1 (indep): Deducing $u \geqslant p$

> *(ii) Given that each particle loses the same (non-zero) amount of kinetic energy in the collision, show that*
> $$u + v + p + q = 0,$$
> *and that, if $m \neq M$,*
> $$e = \frac{(M + 3m)u + (3M + m)v}{(M - m)(u - v)}.$$

The first particle loses an amount of kinetic energy equal to $\frac{1}{2}m(u^2 - p^2)$; the second loses $\frac{1}{2}M(v^2 - q^2)$, so we are given

$$\tfrac{1}{2}m(u^2 - p^2) = \tfrac{1}{2}M(v^2 - q^2),$$

so

$$m(u^2 - p^2) - M(v^2 - q^2) = 0.$$

Again, we use $M(v - q) = m(p - u)$, so that

$$\begin{aligned} m(u^2 - p^2) - M(v^2 - q^2) &= m(u + p)(u - p) - M(v + q)(v - q) \\ &= m(u + p)(u - p) - m(v + q)(p - u) \\ &= m(p + q + u + v)(u - p) \\ &= 0. \end{aligned}$$

Since the amount of kinetic energy lost is non-zero, we have $E > 0$ (in the notation of part (i)), so that $u > p$. Thus we must have $p + q + u + v = 0$.

We can also equate the loss of energy of each particle with $\frac{1}{2}E$, so:

$$\tfrac{1}{2}m(u^2 - p^2) = \tfrac{1}{4}m(u - p)(u - v)(1 - e),$$

giving

$$u + p = \tfrac{1}{2}(u - v)(1 - e). \tag{3}$$

We now need to eliminate p, and rearrange to get an expression for e. Equation (2) gives

$$Mq - Mp = Me(u - v),$$

and subtracting this from (1), $mp + Mq = mu + Mv$, gives

$$(m + M)p = (m - Me)u + (M + Me)v.$$

We multiply (3) by $m + M$ to get:

$$(m + M)p + (m + M)u = \tfrac{1}{2}(M + m)(u - v)(1 - e),$$

and then substitute in our expression for $(m + M)p$ to give us:

$$(m - Me)u + (M + Me)v + (m + M)u = \tfrac{1}{2}(M + m)(u - v)(1 - e).$$

We expand and rearrange to collect terms which are multiples of e:

$$2(mu - Mue + Mv + Mve + mu + Mu) = (Mu - Mv + mu - mv)(1 - e)$$

so

$$(2Mv - 2Mu + Mu - Mv + mu - mv)e = Mu - Mv + mu - mv - 4mu - 2Mv - 2Mu,$$

which leads to

$$(M - m)(v - u)e = -Mu - 3Mv - 3mu - mv.$$

We can right the right hand side as $-(M + 3m)u - (3M + m)v$, so that assuming $M \neq m$, we can divide by $(M - m)(v - u)$ to get

$$e = \frac{(M + 3m)u + (3M + m)v}{(M - m)(v - u)},$$

as we wanted.

Marks

M1: Difference of loss of KE of two particles is zero

M1: Simplifying difference expression using CoM

A1: Deducing factorised expression $m(p+q+u+v)(u-p)$ for (twice) loss of KE

A1 cso: Deducing $p + q + u + v = 0$ (must explicitly explain that $u \neq p$)

M1: Equating loss of KE in one particle with $\frac{1}{2}E$

A1 cao: Deducing expression for $u + p$

M1: Solve original CoM and LoR equations to find expression for p (or $(M + m)p$)

A1: Correct expression for p (or $(M + m)p$)

M1: Substituting p into expression for $u + p$

M1: Rearranging to get $(\cdots)e = (\cdots)$

A1: Simplifying this expression to correct expression for $(M - m)(v - u)e$; can follow through incorrect working for this mark

A1 cso: Deducing stated expression for e

Question 12

Prove that, for any real numbers x and y, $x^2 + y^2 \geqslant 2xy$.

We can rearrange the inequality to get

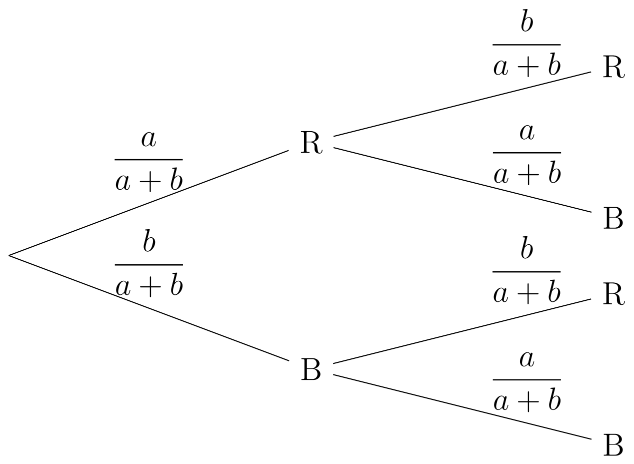
$$x^2 - 2xy + y^2 \geqslant 0.$$

But the left hand side is just $(x-y)^2$, so this inequality becomes $(x-y)^2 \geqslant 0$. This is clearly true, as any real number squared is non-negative.

Marks
M1: Considering $(x-y)^2 \geqslant 0$
A1 cso: Proving inequality

(i) Carol has two bags of sweets. The first bag contains a red sweets and b blue sweets, whereas the second bag contains b red sweets and a blue sweets, where a and b are positive integers. Carol shakes the bags and picks one sweet from each bag without looking. Prove that the probability that the sweets are of the same colour cannot exceed the probability that they are of different colours.

We can draw a tree diagram to represent this situation:



So

$$\begin{aligned}
\text{P(same colour)} &= \frac{a}{a+b} \cdot \frac{b}{a+b} + \frac{b}{a+b} \cdot \frac{a}{a+b} \\
&= \frac{2ab}{(a+b)^2} \\
\text{P(different colours)} &= \frac{a}{a+b} \cdot \frac{a}{a+b} + \frac{b}{a+b} \cdot \frac{b}{a+b} \\
&= \frac{a^2+b^2}{(a+b)^2}.
\end{aligned}$$

Since $a^2 + b^2 \geqslant 2ab$, it follows that the probability that the sweets are of the same colour cannot exceed the probability that they are of different colours, as required.

Marks

M1: Drawing a correctly structured and labelled tree diagram

A1 cao: All probabilities of branches correct on diagram; alternatively, these may be given as counts of possibilities (so $a+b$ times all of the probabilities)

The above two marks may be implied by subsequent working without an actual tree diagram drawn

M1: Determination of probability that both are the same colour

A1 cao: Correct expression for this probability

M1: Determination of probability that they are different colours

A1 cao: Correct expression for this probability

SC: If the candidate uses possibilities for the tree diagram, gives correct probabilities in conclusion, but does not indicate where the denominator $(a+b)^2$ originates, penalise only the first A1 mark.

A1: Using the initial result to deduce required inequality

(ii) Simon has three bags of sweets. The first bag contains a red sweets, b white sweets and c yellow sweets, where a, b and c are positive integers. The second bag contains b red sweets, c white sweets and a yellow sweets. The third bag contains c red sweets, a white sweets and b yellow sweets. Simon shakes the bags and picks one sweet from each bag without looking. Show that the probability that exactly two of the sweets are of the same colour is

$$\frac{3(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2)}{(a+b+c)^3},$$

and find the probability that the sweets are all of the same colour. Deduce that the probability that exactly two of the sweets are of the same colour is at least 6 times the probability that the sweets are all of the same colour.

We argue in the same way:

$$\begin{aligned}
\text{P(exactly 2 same)} &= \text{P(RRW)} + \text{P(RRY)} + \text{P(RWR)}+ \\
&\qquad \text{P(RYR)} + \text{P(WRR)} + \text{P(YRR)}+ \\
&\quad \text{P(WWR)} + \text{P(WWY)} + \text{P(WRW)}+ \\
&\qquad \text{P(WYW)} + \text{P(RWW)} + \text{P(YWW)}+ \\
&\quad \text{P(YYR)} + \text{P(YYW)} + \text{P(YRY)}+ \\
&\qquad \text{P(YWY)} + \text{P(RYY)} + \text{P(WYY)} \\
&= \frac{aba}{(a+b+c)^3} + \frac{abb}{(a+b+c)^3} + \frac{acc}{(a+b+c)^3}+ \\
&\qquad \frac{aac}{(a+b+c)^3} + \frac{bbc}{(a+b+c)^3} + \frac{cbc}{(a+b+c)^3}+ \\
&\quad \frac{bcc}{(a+b+c)^3} + \frac{bcb}{(a+b+c)^3} + \frac{bba}{(a+b+c)^3}+ \\
&\qquad \frac{baa}{(a+b+c)^3} + \frac{aca}{(a+b+c)^3} + \frac{cca}{(a+b+c)^3}+ \\
&\quad \frac{cac}{(a+b+c)^3} + \frac{caa}{(a+b+c)^3} + \frac{cbb}{(a+b+c)^3}+ \\
&\qquad \frac{ccb}{(a+b+c)^3} + \frac{aab}{(a+b+c)^3} + \frac{bab}{(a+b+c)^3} \\
&= \frac{3(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2)}{(a+b+c)^3}.
\end{aligned}$$

More simply, the probability that all three are the same colour is given by

$$\begin{aligned}
\text{P(all 3 same)} &= \text{P(RRR)} + \text{P(WWW)} + \text{P(YYY)} \\
&= \frac{abc}{(a+b+c)^3} + \frac{bca}{(a+b+c)^3} + \frac{cab}{(a+b+c)^3} \\
&= \frac{3abc}{(a+b+c)^3}.
\end{aligned}$$

Now to find the inequality we want, we apply the initial inequality again: we have $a^2 + b^2 \geqslant 2ab$, so $a^2c + b^2c \geqslant 2abc$. Similarly, $b^2 + c^2 \geqslant 2bc$, so $b^2a + c^2a \geqslant 2abc$, and finally, $c^2b + a^2b \geqslant 2abc$. Thus

$$\begin{aligned}
\frac{3(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2)}{(a+b+c)^3} &= \frac{3(a^2c + b^2c + b^2a + c^2a + c^2b + a^2b)}{(a+b+c)^3} \\
&\geqslant \frac{3(2abc + 2abc + 2abc)}{(a+b+c)^3} \\
&= \frac{6(3abc)}{(a+b+c)^3},
\end{aligned}$$

showing that the probability that exactly two of the sweets are of the same colour is at least 6 times the probability that the sweets are all of the same colour.

Marks

M1: Listing the possibilities for exactly two the same

A1 cao: Getting them all correct

M1 (dep): Determining the probabilities for these possibilities

A1: Getting them all correct (do not penalise lack of explanation of denominator $(a+b+c)^3$ *if already penalised earlier; follow through if not all of the possibilities are correct)*

A1 cso: Deducing stated probability

M1: Listing possibilities for all three the same and determining their probabilities

A1 cao: Deducing probability (not necessarily simplified); do not penalise again if $(a+b+c)^3$ *not justified*

M1: Deducing an inequality connecting the two numerators such as $a^2c + b^2c \geqslant 2abc$

M1 (dep): Applying the inequalities to the expressions for the probabilities

A1: Deducing the inequality $3(a^2b + \cdots) \geqslant 18abc$

[The last three marks may be gained for using other valid approaches to proving inequalities, such as using Muirhead's Inequality]

A1 cso: Deducing the result about the probabilities (may be implied by earlier comments or argument)

Question 13

I seat n boys and 3 girls in a line at random, so that each order of the $n+3$ children is as likely to occur as any other. Let K be the maximum number of consecutive girls in the line so, for example, $K = 1$ if there is at least one boy between each pair of girls.

(i) Find $\mathrm{P}(K = 3)$.

There are two equivalent ways to approach this question: either to regard the boys and girls as all distinct, so that there are $(n+3)!$ possible orders, or to regard all boys as indistinguishable and all girls as indistinguishable, so that there are $\binom{n+3}{3}$ possible orders. We use the latter approach here.

We note that, regarding the boys as indistinguishable and the girls as indistinguishable, there are

$$\binom{n+3}{3} = \frac{1}{6}(n+3)(n+2)(n+1)$$

possible arrangements of the students.

If $K = 3$, this means that all three girls are adjacent. So the situation must be that there are r boys, followed by 3 girls, followed by $n - r$ boys, where $r = 0, 1, \ldots, n$, so there are $n + 1$ possibilities.

Thus

$$\begin{aligned}\mathrm{P}(K = 3) &= \frac{n+1}{\frac{1}{6}(n+3)(n+2)(n+1)} \\ &= \frac{6}{(n+2)(n+3)}.\end{aligned}$$

Marks

M1: Argument for total number of permutations of boys and girls (either distinguishable or indistinguishable)

A1 cso: Either correct total for indistinguishable count, or if deduced $(n+3)!$ distinguishable arrangements, then deducing that need to multiply the $n+1$ indistinguishable possibilities below by $3!n!$

B1: $K = 3$ means that all the girls are adjacent

M1: Correct approach to counting the number of possibilities

A1 cao: Deducing there are $n+1$ possibilities (if counting distinguishable permutations, need to multiply all counts by $3!n! = 6\,n!$; this will make things a lot more difficult, though)

A1 cao: Correct probability (allow equivalent expressions)

(ii) Show that

$$\mathrm{P}(K = 1) = \frac{n(n-1)}{(n+2)(n+3)}.$$

Approach 1: Counting explicitly

To have $K = 1$, we must have each pair of girls separated by at least one boy, like this:

$$\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_1}\,\mathrm{G}\,\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_2}\,\mathrm{G}\,\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_3}\,\mathrm{G}\,\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_4}$$

where $r_1 \geqslant 0$, $r_2 > 0$, $r_3 > 0$, $r_4 \geqslant 0$ and $r_1 + r_2 + r_3 + r_4 = n$.

If r_1 and r_2 are fixed, then we must have $r_3 + r_4 = n - (r_1 + r_2)$, so we can have $r_3 = 1$, 2, $\ldots$, $n - (r_1 + r_2)$, giving $n - (r_1 + r_2)$ possibilities for r_3 and r_4.

Thus, if r_1 is fixed, r_2 could be 1, 2, $\ldots$, $n - r_1 - 1$ (but not $n - r_1$, as we must have $r_3 > 0$). Thus the number of possibilities for a fixed value of r_1 is given by

$$\begin{aligned}
\sum_{r_2=1}^{n-r_1-1} (n - r_1 - r_2) &= \sum_{r_2=1}^{n-r_1-1} (n - r_1) - \sum_{r_2=1}^{n-r_1-1} r_2 \\
&= (n - r_1 - 1)(n - r_1) - \tfrac{1}{2}(n - r_1 - 1)(n - r_1) \\
&= \tfrac{1}{2}(n - r_1 - 1)(n - r_1) \\
&= \tfrac{1}{2}(n^2 - 2nr_1 + r_1^2 - n + r_1).
\end{aligned}$$

Now, r_1 can take the values 0, 1, 2, $\ldots$, $n - 2$ (as we need $r_1 > 0$ and $r_2 > 0$), giving the total number of possibilities as

$$\begin{aligned}
&\sum_{r_1=0}^{n-2} \tfrac{1}{2}(n^2 - 2nr_1 + r_1^2 - n + r_1) \\
&\quad= \sum_{r_1=0}^{n-2} \tfrac{1}{2}(n^2 - n) - \tfrac{1}{2}(2n - 1)\sum_{r_1=0}^{n-2} r_1 + \tfrac{1}{2}\sum_{r_1=0}^{n-2} r_1^2 \\
&\quad= \tfrac{1}{2}(n - 1)(n^2 - n) - \tfrac{1}{2}(2n - 1)\cdot\tfrac{1}{2}(n - 2)(n - 1) + \tfrac{1}{12}(n - 2)(n - 1)(2n - 3) \\
&\quad= \tfrac{1}{12}(n - 1)\big(6(n^2 - n) - 3(2n - 1)(n - 2) + (n - 2)(2n - 3)\big) \\
&\quad= \tfrac{1}{12}(n - 1)(6n^2 - 6n - 6n^2 + 15n - 6 + 2n^2 - 7n + 6) \\
&\quad= \tfrac{1}{12}(n - 1)(2n^2 + 2n) \\
&\quad= \tfrac{1}{6}n(n - 1)(n + 1).
\end{aligned}$$

Thus we can finally deduce

$$\begin{aligned}
\mathrm{P}(K = 1) &= \frac{\tfrac{1}{6}n(n - 1)(n + 1)}{\tfrac{1}{6}(n + 3)(n + 2)(n + 1)} \\
&= \frac{n(n - 1)}{(n + 2)(n + 3)}.
\end{aligned}$$

Approach 2: A combinatorial argument

We have to place each of the three girls either between two boys or at the end of the line, and we cannot have two girls adjacent. We can think of the line as n boys with gaps between them and at the ends, like this:

$$\_\,\mathrm{B}\,\_\,\mathrm{B}\,\_\,\ldots\,\_\,\mathrm{B}\,\_\,\mathrm{B}\,\_$$

Note that there are $n+1$ gaps (one to the right of each boy, and one at the left of the line). Three of the gaps are to be filled with girls, giving $\binom{n+1}{3} = \frac{1}{6}(n+1)n(n-1)$ ways of choosing them. Therefore

$$\begin{aligned}
\mathrm{P}(K=1) &= \frac{\frac{1}{6}(n+1)n(n-1)}{\frac{1}{6}(n+3)(n+2)(n+1)} \\
&= \frac{n(n-1)}{(n+2)(n+3)}.
\end{aligned}$$

Approach 3: Another combinatorial argument

We add one more boy at the right end of the line. In this way, we have a boy to the right of each girl, as follows:

$$\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_1}\,\mathrm{GB}\,\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_2}\,\mathrm{GB}\,\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_3}\,\mathrm{GB}\,\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_4}$$

where this time, we have $r_1 \geqslant 0$, $r_2 \geqslant 0$, $r_3 \geqslant 0$ and $r_4 \geqslant 0$. Also, as there are now $n+1$ boys, we have $r_1 + r_2 + r_3 + r_4 = n + 1 - 3 = n - 2$.

So if we think of GB as one 'person', there are $n-2$ boys and 3 GBs, giving $n+1$ 'people' in total. There are $\binom{n+1}{3} = \frac{1}{6}(n+1)n(n-1)$ ways of arranging them, giving

$$\begin{aligned}
\mathrm{P}(K=1) &= \frac{\frac{1}{6}(n+1)n(n-1)}{\frac{1}{6}(n+3)(n+2)(n+1)} \\
&= \frac{n(n-1)}{(n+2)(n+3)}.
\end{aligned}$$

Marks

Approach 1:

M1: Explicitly writing out line of boys and girls and naming number of boys in each section

M1: Conditions on r_i (inequalities and sum with some justification)

M1: Fixing r_1 and r_2, and determining the number of possibilities for r_3 and r_4

M1: Fixing r_1 and summing over all possibilities for r_2

A1: Determining the number of possibilities for fixed r_1

M1: Summing over all possibilities for r_1 and splitting the sum into sums which can be calculated

M1: Simplifying the resulting expression

A1 ft: Simplified expression

A1 cao: Determination of $\mathrm{P}(K=1)$

Approach 3:

M1: Adding extra boy at the end for a useful purpose

M1 (dep): Writing out line in form shown (i.e., with GB combinations)

M1: Condition that all $r_i \geqslant 0$

M1: Determination of $r_1 + r_2 + r_3 + r_4$

M1: Regarding GB as one person to be able to use standard binomials to count possibilities
M1: Counting possibilities using binomial coefficient
A1 ft: Correct binomial coefficient
M1: Determining $\mathrm{P}(K = 1)$
A1 cao: Correct $\mathrm{P}(K = 1)$

(iii) Find $\mathrm{E}(K)$.

We could attempt to determine $\mathrm{P}(K = 2)$ directly, but it is far easier to note that K can only take the values 1, 2 or 3. Thus

$$\begin{aligned}\mathrm{P}(K = 2) &= 1 - \mathrm{P}(K = 1) - \mathrm{P}(K = 3)\\ &= \frac{(n+2)(n+3) - n(n-1) - 6}{(n+2)(n+3)}\\ &= \frac{n^2 + 5n + 6 - n^2 + n - 6}{(n+2)(n+3)}\\ &= \frac{6n}{(n+2)(n+3)}.\end{aligned}$$

Since $\mathrm{E}(K) = \sum_k k.\mathrm{P}(K = k)$, we can now calculate $\mathrm{E}(K)$:

$$\begin{aligned}\mathrm{E}(K) &= 1.\mathrm{P}(K = 1) + 2.\mathrm{P}(K = 2) + 3.\mathrm{P}(K = 3)\\ &= \frac{n(n-1) + 2\cdot 6n + 3\cdot 6}{(n+2)(n+3)}\\ &= \frac{n^2 - n + 12n + 18}{(n+2)(n+3)}\\ &= \frac{n^2 + 11n + 18}{(n+2)(n+3)}\\ &= \frac{(n+2)(n+9)}{(n+2)(n+3)}\\ &= \frac{n+9}{n+3}.\end{aligned}$$

Marks
M1: Determination of $\mathrm{P}(K = 2)$ *either via method given above or otherwise*
A1: Correct $\mathrm{P}(K = 2)$
M1: Correct expression for $\mathrm{E}(K)$ *in terms of* $\mathrm{P}(K = k)$
A1 ft: Substituting in expressions for $\mathrm{P}(K = k)$
A1 cao: Determination of $\mathrm{E}(K)$, *not necessarily fully simplified*